THE SPHERICALLY SYMMETRIC STEFAN PROBLEM WITH A BOUNDARY CONDITION OF THE SECOND KIND

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We consider the spherically symmetric Stefan problem with a boundary condition of the second kind. For the temperature distribution in the region under consideration we obtain a nonlinear integro-differential equation which can be solved by the method of successive approximations. The determination of the time taken for the required boundary to reach a given position is reduced to quadratures.

The problem with a required boundary for the equation of heat conduction finds wide application in the technology of foundry work, in the thermal treatment of metals and in crystal growing. The solution of the problem in a sufficiently general formulation involves great mathematical difficulties. Various approaches are known to the solution of the problem. In [1, 2] the solution of the Stefan problem is reduced to the solution of a functional system of equations; in [3] the application of the two-sided Laplace-Carson transform with respect to a space variable to the solution of the Stefan problem is discussed, as a result of which the functional system of equations becomes identical to the system obtained in [4] by the method of continuing the initial conditions. Very effective in the solution of the Stefan problem is the method of expanding in a series in the "instantaneous" or "local" eingenfunctions of the problem [5-7]. In [8] the method of successive approximations was used to solve the plane single-phase Stefan problem, as is done below for the solution of the spherically symmetric single-phase freezing problem with a boundary condition of the second kind.

The mathematical formulation of the problem takes the following form after the introduction of appropriately chosen dimensionless variables:

$$\frac{\partial u}{\partial t} = \frac{1}{x^2}\frac{\partial}{\partial x}x^2\frac{\partial u}{\partial x}, \quad 1 < x < \Delta(t); \quad 0 < t < M < \infty \tag{1}$$

$$\frac{\partial u}{\partial x} = 1 \qquad (x = 1) \tag{2}$$

$$u = 0 \qquad (x = \Delta(t)) \tag{3}$$

$$\frac{\partial u}{\partial x} = \alpha\frac{d\Delta}{dt} \qquad (x = \Delta(t)) \tag{4}$$

$$\Delta(0) = 1 \tag{5}$$

The problem (1)-(5) corresponds to the external icing of a sphere from the surface of which there is a constant heat flux, while the temperature of the surrounding medium at the initial moment of time is equal to the phase change temperature. A result of solving the problem is the determination of the dimensionless temperature of the medium u(x, t) in the frozen layer and the determination for each moment of time t of the position of the phase change boundary $x = \Delta(t)$ as a function of the dimensionless heat of phase change α.

We integrate (1) with respect to x from 1 to x and take account of the boundary condition (2):

$$\frac{\partial u}{\partial x} = \frac{1}{x^2} + \frac{1}{x^2}\int_1^x x^2\frac{\partial u}{\partial t}dx \tag{6}$$

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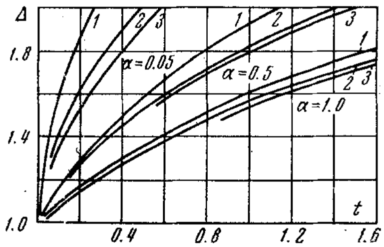

Fig. 1

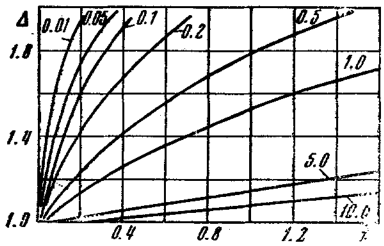

Fig. 2

We write the resulting equation for $x=\Delta(t)$ and note the condition on the required boundary (4). Taking $\Delta=\Delta(t)$ as the independent variable, we obtain

$$\frac{d\Delta}{dt}=\left[\alpha\Delta^2-\int_1^\Delta x^2\frac{\partial u}{\partial\Delta}dx\right]^{-1} \tag{7}$$

Integrating (7) and taking note of the initial condition (5), we obtain an expression for the time taken for the required boundary to reach a given position in space .

$$t=\frac{\alpha}{3}(\Delta^3-1)-\int_1^\Delta\int_1^\Delta x^2\frac{\partial u}{\partial\Delta}dx\,d\Delta \tag{8}$$

Integrating (6) with respect to x from Δ to x, noting the condition on the required boundary (3), and using (7), we obtain the following functional equation:

$$u(x,\Delta)=\frac{1}{\Delta}-\frac{1}{x}+\int_\Delta^x\frac{1}{x^2}\int_1^x x^2\frac{\partial u}{\partial\Delta}dx\,dx\left[\alpha\Delta^2-\int_1^\Delta x^2\frac{\partial u}{\partial\Delta}dx\right]^{-1} \tag{9}$$

which can be solved by the method of successive approximations, as follows:

$$u_0=\frac{1}{\Delta}-\frac{1}{x},\qquad u_{k+1}=u_0+\int_\Delta^x x^2\frac{\partial u_k}{\partial\Delta}dx\,dx\left[\alpha\Delta^2-\int_1^\Delta x^2\frac{\partial u_k}{\partial\Delta}dx\right]^{-1} \tag{10}$$

The zero-order approximation for $u(x,\Delta)$ corresponds to the limiting case of an infinitely small specific heat of the medium, which can be obtained using Leibenzon's method for the approximate solution of the Stefan problem [9], the application of which to the solution of the problem of the solidification of a sphere was discussed by Kovner [10]. Substituting $u_k(x,\Delta)$ in (8), we obtain the approximation for $t_k=t_k(\Delta)$.

The nature of the convergence of the iteration process (10) is easily seen in Fig. 1 where for $\alpha=0.05$, 0.50, and 1.0 the results are given of calculating the zero-order (1), first-order (2), and second-order (3) approximations for the position of the phase change boundary. Figure 2 shows the results of calculating the second approximation for the required boundary for various values of the nondimensional heat of phase change α.

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